Mathematical Toolkit **Spring 2023**

Lecture 7: April 10, 2023

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1 Singular Value Decomposition for matrices

Using the previous discussion, we can write matrices in convenient form. Let $A \in \mathbb{C}^{m \times n}$, which can be thought of as an operator from $\mathbb{C}^n$ to $\mathbb{C}^m$. Let $\sigma_1, \ldots, \sigma_r$ be the non-zero singular values and let $v_1, \ldots, v_r$ and $w_1, \ldots, w_r$ be the right and left singular vectors respectively. Note that $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$ and $v \in V, w \in W$, we can write the operator $|w\rangle \langle v|$ as the matrix wv^*, where v^* denotes $\overline{v^T}$. This is because for any $u \in V$, $wv^*u = w(v^*u) = \langle v, u\rangle \cdot w$. Thus, we can write

$$A = \sum_{i=1}^{r} \sigma_i \cdot w_i v_i^* .$$

Let $W \in \mathbb{C}^{m \times r}$ be a matrix with $w_1, \ldots, w_r$ as columns, such that i^{th} column equals w_i. Similarly, let $V \in \mathbb{C}^{n \times r}$ be a matrix with $v_1, \ldots, v_r$ as the columns. Let $\Sigma \in \mathbb{C}^{r \times r}$ be a diagonal matrix with $\Sigma_{ii} = \sigma_i$. Then, check that the above expression for A can also be written as

$$A = W \Sigma V^* ,$$

where $V^* = \overline{V^T}$ as before.

We can also complete the bases $\{v_1, \ldots, v_r\}$ and $\{w_1, \ldots, w_r\}$ to bases for $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively and write the above in terms of unitary matrices.

Definition 1.1 *A matrix $U \in \mathbb{C}^{n \times n}$ is known as a* unitary matrix *if the columns of U form an orthonormal basis for $\mathbb{C}^n$.*

Proposition 1.2 *Let $U \in \mathbb{C}^{n \times n}$ be a unitary matrix. Then $UU^* = U^*U = \mathrm{id}$, where id denotes the identity matrix.*

Let $\{v_1, \ldots, v_n\}$ be a completion of $\{v_1, \ldots, v_r\}$ to an orthonormal basis of $\mathbb{C}^n$, and let $V_n \in \mathbb{C}^{n \times n}$ be a unitary matrix with $\{v_1, \ldots, v_n\}$ as columns. Similarly, let $W_m \in \mathbb{C}^{m \times m}$ be a unitary matrix with a completion of $\{w_1, \ldots, w_r\}$ as columns. Let $\Sigma' \in \mathbb{C}^{m \times n}$ be a matrix with $\Sigma'_{ii} = \sigma_i$ if $i \leq r$, and all other entries equal to zero. Then, we can also write

$$A = W_m \Sigma' V_n^* .$$

2 Low-rank approximation for matrices

Given a matrix $A \in \mathbb{C}^{m\times n}$, we want to find a matrix B of rank at most k which "approximates" A. For now we will consider the notion of approximation in spectral norm i.e., we want to minimize $\|A-B\|_2$, where

$$\|(A-B)\|_2 \;=\; \max_{v\neq 0} \frac{\|(A-B)v\|_2}{\|v\|_2}\,.$$

Here, $\|v\|_2 \;=\; \sqrt{\langle v, v\rangle}$ denotes the norm defined by the standard inner product on $\mathbb{C}^n$. The 2 in the notation $\|\cdot\|_2$ comes from the expression we get by expressing v in the orthonormal basis of the coordinate vectors. If $v = (c_1,\ldots,c_n)^T$, then $\|v\|_2 \;=\; \left(\sum_{i=1}^n |c_i|^2\right)^{1/2}$ which is simply the Euclidean norm we are familiar with.[1] Note that while the norm here seems to be defined in terms of the coefficients, which indeed depend on the choice of the orthonormal basis, the value of the norm is in fact $\sqrt{\langle v, v\rangle}$ *which is just a function of the vector itself and not of the basis we are working with*. The basis and the coefficients merely provide a convenient way of computing the norm.

SVD also gives the optimal solution for another notion of approximation: minimizing the Frobenius norm $\|A-B\|_F$, which equals $(\sum_{ij}(A_{ij}-B_{ij})^2)^{1/2}$. We will see this later. Let $A = \sum_{i=1}^r w_i v_i^*$ be the singular value decomposition of A and let $\sigma_1 \geq \cdots \geq \sigma_r > 0$. If $k \geq r$, we can simply use $B = A$ since $\mathrm{rank}(A) = r$. If $k < r$, we claim that $A_k = \sum_{i=1}^k \sigma_i w_i v_i^*$ is the optimal solution.

Proposition 2.1 $\|A - A_k\|_2 \;=\; \sigma_{k+1}$.

Proof: Complete $v_1,\ldots,v_k$ to an orthonormal basis $v_1,\ldots,v_n$ for $\mathbb{C}^n$. Given any $v \in \mathbb{C}^n$, we can uniquely express it as $\sum_{i=1}^n c_i \cdot v_i$ for appropriate coefficients $c_1,\ldots,c_n$. Thus, we have

$$(A-A_k)v \;=\; \left(\sum_{j=k+1}^r \sigma_j \cdot w_j v_j *\right)\left(\sum_{i=1}^n c_i \cdot v_i\right) \;=\; \sum_{j=k+1}^r \sum_{i=1}^n c_i\sigma_j \cdot \langle v_j, v_i\rangle \cdot w_j \;=\; \sum_{j=k+1}^r c_j\sigma_j \cdot w_j\,,$$

where the last equality uses the orthonormality of $\{v_1,\ldots,v_n\}$. We can also complete $w_1,\ldots,w_r$ to an orthonormal basis $w_1,\ldots,w_m$ for $\mathbb{C}^m$. Since $(A-A_k)$ is already expressed in this basis above, we get that

$$\|(A-A_k)v\|_2^2 \;=\; \left\|\sum_{j=k+1}^r c_j\sigma_j \cdot w_j\right\|_2^2 \;=\; \left\langle \sum_{j=k+1}^r c_j\sigma_j \cdot w_j, \sum_{j=k+1}^r c_j\sigma_j \cdot w_j\right\rangle \;=\; \sum_{j=k+1}^r |c_j|^2 \cdot \sigma_j^2\,.$$

[1] In general, one can consider the norm $\|v\|_p := \left(\sum_{i=1}^n |c_i|^p\right)^{1/p}$ for any $p \geq 1$. While these are indeed valid notions of distance satisfying a triangle inequality for any $p \geq 1$, they do not arise as a square root of an inner product when $p \neq 2$.

Finally, as in the computation with Rayleigh quotients, we have that for any $v \neq 0$ expressed as $v = \sum_{i=1}^{n} c_i \cdot v_i$,

$$\frac{\|(A - A_k)v\|_2^2}{\|v\|_2^2} \;=\; \frac{\sum_{j=k+1}^{r} |c_j|^2 \cdot \sigma_j^2}{\sum_{i=1}^{n} |c_i|^2} \;\leq\; \frac{\sum_{j=k+1}^{r} |c_j|^2 \cdot \sigma_{k+1}^2}{\sum_{i=1}^{n} |c_i|^2} \;\leq\; \sigma_{k+1}^2 \,.$$

This gives that $\|A - A_k\|_2 \leq \sigma_{k+1}$. Check that it is in fact equal to σ_{k+1} (why?) ■

In fact the proof above actually shows the following:

Exercise 2.2 *Let $M \in \mathbb{C}^{m \times n}$ be any matrix with singular values $\sigma_1 \geq \cdots \sigma_r > 0$. Then, $\|M\|_2 = \sigma_1$ i.e., the spectral norm of a matrix is actually equal to its largest singular value.*

Thus, we know that the error of the best approximation B is at most σ_{k+1}. To show the lower bound, we need the following fact.

Exercise 2.3 *Let V be a finite-dimensional vector space and let S_1, S_2 be subspaces of V. Then, $S_1 \cap S_2$ is also a subspace and satisfies*

$$\dim(S_1 \cap S_2) \;\geq\; \dim(S_1) + \dim(S_2) - \dim(V) \,.$$

We can now show the following.

Proposition 2.4 *Let $B \in \mathbb{C}^{m \times n}$ have $\mathrm{rank}(B) \leq k$ and let $k < r$. Then $\|A - B\|_2 \geq \sigma_{k+1}$.*

Proof: By rank-nullity theorem $\dim(\ker(B)) \geq n - k$. Thus, by the fact above

$$\dim\left(\ker(B) \cap \mathrm{Span}\left(v_1, \ldots, v_{k+1}\right)\right) \;\geq\; (n-k) + (k+1) - n \;\geq\; 1 \,.$$

Thus, there exists a $z \in \ker(B) \cap \mathrm{Span}\left(v_1, \ldots, v_{k+1}\right) \setminus \{0\}$. Then,

$$\begin{aligned} \|(A - B)z\|_2^2 \;=\; \|Az\|_2^2 \;=\; \langle z, A^*Az \rangle \;&=\; \mathcal{R}_{A^*A}(z) \cdot \|z\|_2^2 \\ &\geq\; \left(\min_{y \in \mathrm{Span}(v_1, \ldots, v_{k+1}) \setminus \{0\}} \mathcal{R}_{A^*A}(y) \right) \cdot \|z\|_2^2 \\ &\geq\; \sigma_{k+1}^2 \cdot \|z\|_2^2 \,. \end{aligned}$$

Thus, there exists a $z \neq 0$ such that $\|(A - B)z\|_2 \geq \sigma_{k+1} \cdot \|z\|_2$, which implies $\|A - B\|_2 \geq \sigma_{k+1}$. ■